The field axioms

***Closure* of *F* under addition and multiplication**

For all a, b in F, both $a + b$ and $a \cdot b$ are in F (or more formally, + and · are binary operations on F).

***Associativity* of addition and multiplication**

For all a, b, and c in F, the following equalities hold: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

***Commutativity* of addition and multiplication**

For all a and b in F, the following equalities hold: $a + b = b + a$ and $a \cdot b = b \cdot a$.

Existence of additive and multiplicative *identity elements*

There exists an element of F, called the *additive identity* element and denoted by 0, such that for all a in F, $a + 0 = a$. Likewise, there is an element, called the *multiplicative identity* element and denoted by 1, such that for all a in F, $a \cdot 1 = a$. To exclude the trivial ring, the additive identity and the multiplicative identity are required to be distinct.

Existence of *additive inverses* and *multiplicative inverses*

For every a in F, there exists an element $-a$ in F, such that $a + (-a) = 0$. Similarly, for any a in F other than 0, there exists an element a^{-1} in F, such that $a \cdot a^{-1} = 1$. (The elements $a + (-b)$ and $a \cdot b^{-1}$ are also denoted $a - b$ and a/b, respectively.) In other words, *subtraction* and *division* operations exist.

***Distributivity* of multiplication over addition**

For all a, b and c in F, the following equality holds: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Proposition 1.14 from Rudin. The axioms for addition imply the following statements.

(a) If $x + y = x + z$ then $y = z$ (cancellation)
(b) If $x + y = x$ then $y = 0$ (uniqueness of the additive identity)
(c) If $x + y = 0$ then $y = -x$ (uniqueness of the additive inverse)
(d) $-(-x) = x$ (double negation)

Proof. If $x + y = x + z$, the axioms for addition give

$$y = 0 + y = (-x + x) + y = -x + (x + y) = -x + (x + z) = (-x + x) + z = 0 + z = z.$$

This proves (a). Take $z = 0$ in (a) to obtain (b). Take $z = -x$ in (a) to obtain (c). Since $-x + x = 0$, (c) (with $-x$ in place of x) gives (d). ∎

Proposition 1.15 from Rudin. The axioms for multiplication imply the following statements.

(a) If $x \neq 0$ and $xy = xz$ then $y = z$ (cancellation)
(b) If $x \neq 0$ and $xy = x$ then $y = 1$ (uniqueness of the multiplicative identity)
(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$ (uniqueness of the multiplicative inverse)
(d) If $x \neq 0$ then $1/(1/x) = x$.

Proposition 1.16 from Rudin. The field axioms imply the following statements, for any x, y, and $z \in F$.

(a) $0x = 0$.
(b) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.
(c) $(-x)y = -(xy) = x(-y)$.
(d) $(-x)(-y) = xy$

Proof. $0x + 0x = (0 + 0)x = 0x$. Hence 1.14(b) implies that $0x = 0$, and (a) holds. Next, assume $x \neq 0$, $y \neq 0$, but $xy = 0$. Then (a) gives

$$1 = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)xy = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)0 = 0$$

a contradiction. Thus (b) holds. The first equality in (c) comes from

$$(-x)y + xy = (-x + x)y = 0y = 0,$$

combined with 1.14(c); the other half of (c) is proved in the same way. Finally,

$$(-x)(-y) = -[x(-y)] = -[-(xy)] = xy$$

by (c) and 1.14(d). ∎